$$\times\left(1+\sin\frac{\theta}{2}\right)\Big]+\frac{\partial}{\partial\theta}\operatorname{Re}\frac{S+T}{2}\int\limits_0^1\frac{y^{1-n_1}-1}{y^2}\left[\frac{1}{\sqrt{y^2-2y\cos\theta+1}}-1-y\cos\theta\right]dy$$

It is easy to see that the integrands are bounded everywhere for $0\leqslant\theta\leqslant\pi$, $0\leqslant y\leqslant 1$. The first two members in the relationships (8) agree asymptotically, for $\theta\to 0$, with the isolated singularity in the solutions in [1, 3].

Example 2. Evidently the solution of the problem of compression of a sphere by concentrated forces applied to its poles $r=R$, $\theta=0$ and $\theta=\pi$ is the superposition of two solutions of type (7), namely

$$u_r(r,\theta)=u_r^*(r,\theta)+u_r^*(r,\pi-\theta)$$
$$u_\theta(r,\theta)=u_\theta^*(r,\theta)-u_\theta^*(r,\pi-\theta)$$

Thus the solution of the problem of deformation of a sphere by an axisymmetric normal loading is represented by the quadratures of (4), (5). The advantage of this representation will be that it is valid even for loadings having a strong discontinuity of a concentrated force type.

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Translated by M. D. F.

CERTAIN TYPE OF INTEGRAL EQUATIONS APPEARING IN CONTACT PROBLEMS OF THE THEORY OF ELASTICITY

PMM Vol. 33, №6, 1969, pp. 1034-1041

V. A. BABESHKO
(Rostov-on-Don)

(Received May 29, 1969)

A method of investigating the integral equation for the case when its kernel is a meromorphic function with simple poles and double zeros, is presented. The integral equation is reduced to an infinite system of linear algebraic equations which normally has a solution, and this solution is constructed together with that of a certain finite system. A general form of sufficient conditions which must be imposed on the right side of the equation to ensure that it has a unique solution, is derived.

Mixed problems of the theory of elasticity on determination of stresses generated under a die impressed into an elastic layer lying without friction on a rigid foundation [1], and the problem concerning the stresses generated under a wheel with a tyre, fitted on an elastic shaft [2], both lead to an integral equation of the form

$$\mathbf{K}q \equiv \int_{-a}^{a} k(x-\xi)\, q(\xi)\, d\xi = \pi\varphi(x), \qquad |x| \leqslant a \tag{0.1}$$

The kernel $k(t)$ of the integral equation for the above problems is given, respectively, by

$$k(t) = \int_0^\infty K(u) \cos ut\, du, \qquad K(u) = \frac{2\,\mathrm{sh}^2 u}{u(\mathrm{sh}\, 2u + 2u)} \tag{0.2}$$

$$K(u) = \frac{I_1^2(u)}{u^2[I_0^2(u) - I_1^2(u)] - 2(1-\sigma) I_1^2(u)} \qquad (0 < \sigma < 0.5)$$

where $I_0(z)$ and $I_1(z)$ denote modified Bessel functions of the zero and the first order. Equation (0.1) is dealt with in [1–4] e. a. If $K(u)$ is a rational function, a solution can, generally, be constructed for (0.1) in a finite form; this was established in [4]. Paper [5] proposes a method of constructing an asymptotic solution for (0.1) at $a \to \infty$, under the assumption that $K(u)$ is a meromorphic function with simple poles and zeros. This method, however, beaks down when $K(u)$ has multiple zeros.

1. We shall assume that $K(z)$ is an even function, real on the real axis and meromorphic on the complex plane, with double zeros and simple poles not lying on the real axis and represented by the following asymptotic expression on the upper semiplane

$$z_n = i(\beta n + b) \pm c_1 \ln n + O(n^{-1}\ln n) \qquad (|z_n| \leqslant |z_{n+1}|) \tag{1.1}$$
$$\zeta_n = i(\beta n + g) \pm c_2 \ln n + O(n^{-1}\ln n) \qquad (|\zeta_n| \leqslant |\zeta_{n+1}|) \quad (n \to \infty)$$

The constants entering (1.1) are real.

We also assume that $K(z)$ has the following asymptotic property

$$K(z) = c^2 z^{-1}[1 + \alpha z^{-1} + O(z^{-2})], \quad |z| \to \infty, \quad |\arg z \pm \pi/2| > \varepsilon > 0 \tag{1.2}$$

We shall consider the functions of the form

$$K_+(z) = \sqrt{A} \lim_{n\to\infty} \prod_{k=1}^{2n} \left(1 + \frac{z}{z_k}\right)\left(1 + \frac{z}{\zeta_k}\right)^{-1}, \quad K_-(z) = K_+(-z), \quad A = K(0) \tag{1.3}$$

Each zero appearing in the product (1.3) is a double one, and we can easily see that

$$K_+(z) K_-(z) = K(z) \tag{1.4}$$

Lemma 1.1. The following asymptotic estimates are correct:

$$K_\pm(z) = c z^{-0.5}[1 + o(1)], \quad |z| \to \infty, \quad |\arg z \pm \pi/2| > \varepsilon > 0$$

At large $|z|$ the function $K_+(z)$ can be approximated by a rational combination of the Euler's gamma functions, hence the proof follows from (1.2).

The requirement that relations of the form

$$H_+(-z_k) = ck^{-1-\gamma_1}[1 + o(1)], \quad [H_+^{-1}(-z_k)]' = ck^{\gamma_2}[1 + o(1)] \quad (k \to \infty)$$
$$0 < \gamma_1, \gamma_2 \leqslant 0.5 \tag{1.5}$$

hold for the expressions

$$H_+(-z_k) = \lim_{z\to -z_k} \frac{K_+(z)}{(z + z_k)^2}, \qquad H_+'(-z_k) = \lim_{z\to -z_k} \frac{d}{dz}\left\{\frac{K_+(z)}{(z + z_k)^2}\right\} \tag{1.6}$$

represents the last condition to be imposed on the function $K(z)$.

In the following we shall make use of certain information from the theory of integral equations of the first kind on a semiaxis, which have the form

$$\int_0^\infty k(x-\xi)\,\alpha(\xi)\,d\xi = \pi f(x) \qquad (0 < x < \infty) \tag{1.7}$$

If the kernel $k(t)$ satisfies the above conditions, the results of [6, 7] can be used to prove the following theorem.

Theorem 1.1. Equation (1.7) has a unique solution in the class of functions $\alpha(x)$ such, that $\alpha(x)x^{0,5}\exp(-\mu x) \in C_0{}^{\circ}(0,\infty), \qquad 0<\mu<\inf(\operatorname{Im}\zeta_1, \operatorname{Im} z_1)$

for any right side of $f(x)$ possessing the property (1.8)

$$f'(x)x^{\epsilon}\exp(-\mu x) \in C_0{}^{\lambda}(0,\infty) \qquad (0.5<\lambda-\epsilon<1) \tag{1.9}$$

Here $C_k{}^{\lambda}(a, b)$ denotes a set of functions whose kth order derivatives satisfy the Hölder condition with the index λ, on $[a, b]$.

To prove this theorem we construct such normalizers [7] of (1.7) in the functional $\varphi(x)$-space, that $x^{\lambda}\varphi'(x) \in C_0{}^{\lambda}(0, \infty)$. In addition, we establish in advance that the properties of the kernel described above are sufficient to regard the present space as a Krein space, in which some integral equation of the second kind on a semiaxis [6, 7] normally has a solution.

2. The main result of the present paper follows.

Theorem 2.1. Equation (0.1) has a unique solution in $L_p(-a, a)$ $(p>1)$ for any right part $f(x) \in C_1{}^{\lambda}(-a, a)$, $\lambda > 0.5$; this solution can be written in the form

$$q(x) = \int_{-\infty}^{\infty} \frac{\Phi(\eta)}{K(\eta)} e^{i\eta x}\,d\eta + \sum_{l=1}^{\infty} \{[A_l{}^{+} + (a+x)B_l{}^{+}]\exp iz_l(a+x) +$$

$$+[A_l{}^{-} + (a-x)B_l{}^{-}]\exp iz_l(a-x)]\}, \qquad \varphi(x) = \int_{\infty}^{\infty} \Phi(\eta)\,e^{i\eta x}\,d\eta \tag{2.1}$$

Exact values of the coefficients $A_l^{\pm}$ and $B_l^{\pm}$ are obtained together with the solution of a certain finite system of algebraic equations. In addition, $q(x)$ has the property

$$q(x)(a^2-x^2)^{0.5} \in C_0{}^{\circ}(-a, a) \tag{2.2}$$

Proof of this theorem will be preceded by a number of lemmas and theorems.

In the following, $c(\sigma)$ (where σ is an arbitrary real number) will denote a space of complex sentences $\{x_l\} = X$ possessing the property

$$\sup_l |l^{\sigma}x_l| < \infty, \qquad \lim_{l\to\infty} l^{\sigma}x_l = 0 \tag{2.3}$$

Obviously, $c(\sigma)$ is a Banach space provided that its norm is defined by

$$\|X\|_{c(\sigma)} = \sup_l |l^{\sigma}x_l| \tag{2.4}$$

Moreover, $H(-a, a)$ will denote a space of functions $q(x)$ with the norm given by

$$\|q(x)\|_H = \Big(\int_{-\infty}^{\infty} K(u)\,|Q(u)|^2\,du\Big)^{1/2} < \infty, \qquad Q(u) = \int_{-a}^{a} q(x)\,e^{iux}\,dx \tag{2.5}$$

We note the following inclusion:

$$L_p(-a, a) \subset H(-a, a), \qquad 1<p\leqslant 2 \tag{2.6}$$

It can easily be confirmed that K is an operator, positive definite in $H(-a, a)$. This enables us to prove the following assertion using the familiar methods [8].

Lemma 2.1. Equation (1.1) has a unique solution in $H(-a,a)$ if and only if the inequality

$$\int_{-\infty}^{\infty} |\Phi(u) K_{+}^{-1}(u)|^2 du < \infty \tag{2.7}$$

holds.

It can be shown that (2.7) holds, if

$$\varphi' x (a^2 - x^2)^{\mu} \in C_0^{\lambda}(-a,a), \qquad 0 \leqslant \mu < \lambda < 1 \tag{2.8}$$

Lemma 2.2. Series in (2.1) belong, as functions of x, to

$$L_p(-a,\ a),\ 1 < p < (1-\sigma)^{-1}, \qquad 0 < \sigma < 1 \tag{2.9}$$

provided that the properties

$$\{A_l^{\pm}\} \in c(\sigma), \qquad \{B_l^{\pm}\} \in c(\sigma - 1), \qquad 0 < \sigma < 1 \tag{2.10}$$

take place.

The lemma is easily proved by applying the Minkowski inequality to the infinite series (2.1).

We shall seek the solution $q(x)$ of (0.1) belonging to $L_p(-a,\ a)$ $1 < p \leqslant 2$ in the form of a series (2.1), whose coefficients have the property (2.10).

Theorem 2.1. For $\varphi(x) \in C_1^{\lambda}(-a,\ a)$ $(\lambda > 0.5)$ the equation (0.1) is equivalent to an infinite system of algebraic equations of the form

$$\sum_{l=1}^{\infty} \left\{ \left(\frac{1}{\zeta_r - z_l} \pm \frac{\exp 2aiz_l}{\zeta_r + z_l} \right) x_l^{\pm} + \left[\frac{1}{(\zeta_r - z_l)^2} \pm \left(\frac{1}{\zeta_r + z_l} + 2a \right) \frac{\exp 2aiz_l}{\zeta_r + z_l} \right] y_l^{\pm} \right\} = d_r^{\pm} \tag{2.11}$$

$$x_l^{\pm} = A_l^{+} \pm A_l^{-}, \qquad d_r^{\pm} = \int_{-\infty}^{\infty} \frac{\Phi(\eta)}{K(\eta)} \left(\frac{e^{-i\eta a}}{\eta - \zeta_r} \mp \frac{e^{i\eta a}}{\eta + \zeta_r} \right) d\eta \qquad (r = 1, 2, \ldots)$$
$$y_l^{\pm} = B_l^{+} \pm B_l^{-},$$

$$\{x_l^{\pm}\} \in c(\gamma), \qquad \{y_l^{\pm}\} \in c(\gamma - 1), \qquad \gamma < 0.5 \tag{2.12}$$

We begin by establishing the theorem for the case $\varphi(x) = e^{i\eta x}$. The infinite system is constructed in the following manner. The kernel $k(x)$ is represented in the form of a series by computing the integral (0.2) by the method of residues. The resulting series converges uniformly everywhere except at the point $x = 0$, where $k(x)$ has a logarithmic singularity.

After this we insert the kernel $k(x)$ and the solution $q(x)$ taken in the form (2.1), with $\Phi(\eta) = \delta(\xi - \eta)$ ($\delta(t)$ is the Dirac delta function), into Eq. (0.1) and integrate the result. By (1.1) the series (2.1) converges uniformly on any segment belonging to $(-a,\ a)$. The points $\pm a$ may be found to be singular, but by Lemma 2.2 they are integrable.

This makes possible the assertion that the Dirichlet series obtained by integration, will converge uniformly for all $|x| \leqslant a$.

In addition, when conditions (2.10) and (1.1) hold, this series turns out to be a function which can be analytically continued into the square $|x| \leqslant a$, $|y| \leqslant M$ ($z = x + iy$, M is an arbitrary fixed number).

Transition from the Dirichlet series to the infinite system is carried out on the basis of the following result established by Leont'ev [9], (p. 133). Let the complex numbers ζ_l be zeros of the entire function $P(z)$, whose growth indicatrix is

$$h(\varphi)=\sigma|\sin\varphi|,\ h(\varphi)=\overline{\lim_{\rho\to\infty}}\frac{\ln|P(\rho e^{i\varphi})|}{\rho}$$

Then, if the series

$$Q(z)=\sum_{k=1}^{\infty}[a_k\exp\zeta_k z+b_k\exp(-\zeta_k z)]$$

converges uniformly in the region ω containing the whole of the segment $x=0$, $|y|\leqslant\sigma$, the condition $Q(z)\equiv 0$ in ω implies that all $a_k=b_k=0$. By Lemma 2.1, the same fact can be used to establish the uniqueness of the solution of the system (2.11) in the space (2.12). Transition from the infinite system to the integral equation is effected in the reverse order.

Lemma 2.3. Let

$$x^{\varepsilon}f'(x)\in C_0^{\lambda}(0,\infty),\quad f(x)\equiv 0 \quad \text{for } x\geqslant N>0, \qquad 0.5<\lambda-\varepsilon<1$$

(N is an arbitrary fixed number)

Then the integral equation (1.7) investigated in $L_p(0,\infty)$ $(1<p\leqslant 2)$ is equivalent to the following infinite system:

$$\sum_{l=1}^{\infty}\left[\frac{D_l}{\zeta_r-z_l}+\frac{G_l}{(\zeta_r-z_l)^2}\right]=\delta_r,\qquad \delta_r=\int_{-\infty}^{\infty}\frac{F(\eta)\,d\eta}{K(\eta)(\eta-\zeta_r)} \tag{2.13}$$

in the class of sequences

$$\{D_l\}\in c(\gamma),\qquad \{G_l\}=c(\gamma-1)\qquad (0<\gamma<0.5)$$

Solution of (1.7) has the form (2.14)

$$\alpha(x)=\int_{-\infty}^{\infty}\frac{F(\eta)}{K(\eta)}e^{i\eta x}d\eta+\sum_{l=1}^{\infty}(D_l+xG_l)\exp iz_lx,\quad f(x)=\int_{-\infty}^{\infty}F(\eta)e^{i\eta x}d\eta$$

To prove the direct statement let us solve Eq. (1.7) and represent it in the form (2.14), with

$$D_l=[H_+^{-1}(-z_l)]'\int_{-\infty}^{\infty}\frac{F(\eta)\,d\eta}{(\eta-z_l)K_+(\eta)}-\frac{1}{H_+(-z_l)}\int_{-\infty}^{\infty}\frac{F(\eta)\,d\eta}{(\eta-z_l)^2K_+(\eta)}$$

$$G_l=\frac{1}{H_+(-z_l)}\int_{-\infty}^{\infty}\frac{F(\eta)\,d\eta}{(\eta-z_l)K_+(\eta)} \tag{2.15}$$

Conditions imposed on $f(x)$ in the statement of the lemma are sufficient for $F(\eta)$ to have the following property:

$$F(\eta)=O(\eta^{-1-\lambda+\varepsilon})\qquad (\eta\to\infty) \tag{2.16}$$

Then, taking into account (1.5), we obtain the estimates

$$D_l\sim O(l^{\rho-1}),\quad G_l=O(l^{\rho}),\quad \rho=\sup(\gamma_1,\gamma_2)\leqslant 0.5\qquad (l\to\infty) \tag{2.17}$$

from which (2.13) follows. The converse is proved in a similar manner.

3. Let us write the system (2.11) in the matrix form

$$[A+B(a)]S=D \tag{3.1}$$

where the following notation is employed

$$A=\{a_{rl}\},\qquad B(a)=\{b_{rl}\},\qquad S=\{s_l\},\qquad D=\{d_r^{\pm}\} \tag{3.2}$$

$$a_{r,2l-1}=(\zeta_r-z_l)^{-1},$$

$$b_{r,2l} = \pm z_l (\zeta_r + z_l)^{-1} [(\zeta_r + z_l)^{-1} + 2a] \exp 2aiz_l, \quad s_{2l-1} = x_l \pm \qquad \text{(cont.)}$$

$$a_{r,2l} = (\zeta_r - z_l)^{-2} z_l, \quad b_{r,2l-1} = \pm (\zeta_r + z_l) \exp 2aiz_l, \quad s_{2l} = z_l^{-1} y_l \pm$$

By (2.12) we obviously have

$$S \in c(\gamma), \quad \gamma < 0.5$$

Lemma 3.1. System (3.1) is equivalent to a system of the form

$$S = -A^{-1}B(a)S + D_1, \qquad D_1 = A^{-1}D \tag{3.3}$$

where $A^{-1}B(a)$ is an operator which is fully continuous in $c(\gamma)$, $\gamma < 0.5$. A^{-1} is a bilateral operator inverse to A. The following notation is introduced:

$$A^{-1} = \{\tau_{lr}\}, \qquad \tau_{2l,r} = \{H_+(-z_l)[K_-^{-1}(\zeta_r)]'(\zeta_r - z_l)\}^{-1} \tag{3.4}$$

$$\tau_{2l-1,r} = [H_+^{-1}(-z_l)]' \{[K_-^{-1}(\zeta_r)]'(\zeta_r - z_l)\}^{-1} -$$

$$- \{H_+(-z_l)[K_-^{-1}(\zeta_r)]'(\zeta_r - z_l)^2\}^{-1}$$

$$-A^{-1}B(a) = \{\varepsilon_{lm}\}, \quad \varepsilon_{2l,m} = \pm \frac{[(z_m + z_l)^{-2} + 2a]\, z_m \exp 2aiz_m K_+(z_m)}{z_l (z_m + z_l) H_+(-z_l)}$$

$$\varepsilon_{2l-1,m} = \pm \left\{[H_+^{-1}(-z_l)]' - \frac{H_+^{-1}(-z_l)}{(\zeta_m + z_l)}\right\} \frac{\exp 2aiz_m K_+(z_m)}{(z_m + z_l)} \tag{3.5}$$

Elements of the matrix A^{-1} are constructed by solving [5] a sequence of Eqs. (1.7) on a semiaxis, whose right side $f(x) = \exp i\zeta_r x$, $0 \leqslant x < \infty$. A direct check together with the relations (1.5) show, that the matrices A and A^{-1} commute. Relations (1.5) are also employed to check the operator $A^{-1}B(a)$ for complete continuity in the space $c(\gamma)$, $\gamma < 0.5$.

From (3.5) and (1.5) we obtain the following estimate

$$\varepsilon_{n,m} = O[n^{\gamma-1} m^2 \exp(-2a\beta m)] \qquad (n, m \to \infty) \tag{3.6}$$

Lemma 3.2. System (3.3) is equivalent to a finite system of the following linear algebraic equations possessing a unique solution

$$s_n = \sum_{k=1}^{N} \varepsilon_{n,k}^{\circ} s_k + d_n^{\circ} \qquad (n = 1, \ldots, N) \tag{3.7}$$

To prove this lemma, we shall use a system of the form

$$s_n = \sum_{k=N+1}^{\infty} \varepsilon_{n,k} s_k + d_n^*, \quad d_n^* = d_n + \sum_{k=1}^{N} \varepsilon_{n,k} s_k \quad (n = N+1, \ldots) \tag{3.8}$$

where N is selected from the condition

$$\sup_{n>N} \sum_{k=N+1}^{\infty} |\varepsilon_{n,k}| n^{\sigma} \leqslant q < 1 \qquad (\sigma < 0.5) \tag{3.9}$$

The latter is completely feasible by virtue of the estimate (3.6). But then, system (3.8) will have a unique solution in $c(\sigma)$ for any element d_n^* of this space. We can write this solution in the form

$$S_N = \sum_{m=0}^{\infty} \mathrm{E}_N{}^m D_N^*, \quad S_N = \{s_n\}, \tag{3.10}$$

$$D_N^* = \{d_n^*\} \quad n = N+1, \ldots \quad \mathrm{E}_N = \{\varepsilon_{nk}\} \quad (n, k = N+1, N+2, \ldots), \qquad \mathrm{E}^2 = \mathrm{EE}$$

Inserting (3.10) into the first N equations of (3.3) and taking the expression for d_n^* from (3.8) into account we arrive at the system (3.7), in which

$$d_n^\circ = d_n + \sum_{s=N+1}^{\infty} \sum_{r=N+1}^{\infty} \varepsilon_{ns} u_{sr}^\circ d_r, \qquad \{u_{nr}^\circ\} = \sum_{m=0}^{\infty} \mathrm{E}_N^m$$
$$\varepsilon_{nk}^\circ = \varepsilon_{nk} + \sum_{s=N+1}^{\infty} \sum_{r=N+1}^{\infty} \varepsilon_{ns} u_{sr}^\circ \varepsilon_{rk} \qquad (n = 1, \ldots, N) \tag{3.11}$$

Unique solvability of the system (3.7) follows from the easily perceived equivalence of the latter system and of the integral equation (0.1) which, by Lemma 2.1 has a unique solution. The lemma is proved.

In order to complete the proof of Theorem 2.1 we must establish the property (2.2). Let us write the system (3.1) in the form

$$AS = G, \quad G = D - B(a)S \tag{3.12}$$

and show that Eq. (1.7) on the semiaxis, the right side of which $f(x) = f_1(x) + f_2(x)$ satisfies (1.9), is equivalent to (3.12). Then, comparing the series (2.1) and (2.14) we obtain condition (2.2) by virtue of Theorem 1.1.

Function $f_1(x)$ satisfying the conditions of the Lemma 2.3 (and hence the condition (1.9)), is easily constructed in terms of an element of D. Function $f_2(x)$ corresponding to the element $B(a)\ S$, has the form

$$f_2(x) = \int_{-\infty}^{\infty} F_2(\eta) e^{i\eta x} d\eta, \qquad F_2(\eta) = K(\eta) \sum_{k=1}^{\infty} \frac{z_k s_k e^{2iaz_k}}{\eta + z_k} \tag{3.13}$$

Taking into account (2.13) we may conclude that $f_2(x)$ satisfies the condition (1.9), and this completes the proof of Theorem 2.1.

Thus, in order to construct the solution of (0.1) we must solve the finite system (3.7) (which has a unique solution), then determine the coefficients $A_e^{\pm}$ and $B_e^{\pm}$ using formulas (3.8), (3.2) and (2.11) and insert them into (2.1). From Theorem 2.1 it follows that the solution $q(x)$ of (0.1) has, in general, a singularity of the type $(a^2 - x^2)^{-0.5}$ at the points $x = \pm a$.

Note. The method of investigating equations of the form (0.1) proposed in the present paper is applicable whenever the function $K(u)$ is positive on the real axis, can be expanded into (1.2) and admits the representation

$$K(z) = P_1(z) P_2^{-1}(z)$$

Here $P_k(z)$ are entire functions whose zeros possess the asymptotic representation (1.1) and whose growth indicatrices $h_k(\varphi)$ have the form

$$h_k(\varphi) = \sigma_k |\sin \varphi|$$

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Translated by L. K.

AN INTEGRAL EQUATION AND ITS APPLICATION TO CONTACT PROBLEMS IN THE THEORY OF ELASTICITY WITH FRICTION AND COHESION FORCES

PMM Vol. 33, №6, 1969, pp. 1042-1050
A. S. SOLOV'EV
(Rostov-on-Don)
(Received March 12, 1969)

A singular integral equation is examined. This equation is generated by some mixed problems of the plane theory of elasticity, in particular by problems dealing with the contact between two bodies when friction or complete cohesion are present in the contact region. General properties of the solution of this equation are investigated. The initial singular equation is reduced to Fredholm's integral equation of the second kind through application of regularization by means of the solution of the characteristic equation [1]. For the condition where the kernel is small the resolvent is found for Fredholm's integral equation of the second kind.

Problems of interaction between a stamp and an elastic isotropic strip are examined: displacement of the stamp in the presence of friction between the stamp and the strip, and the impression of the stamp into the strip in case of complete cohesion in the region of contact (*). Solutions of these problems are obtained in the form of power series of a dimensionless small parameter which characterizes the relative length of the contact region. Boundaries for uniform and absolute convergence of these series are established. Examples are presented.

1. Let us examine the following singular integral equation:

$$\frac{\theta}{2}\int_{-1}^{1}\varphi(\xi)\operatorname{sgn}(x-\xi)\,d\xi-\frac{1}{\pi}\int_{-1}^{1}\varphi(\xi)\ln\mu\,|x-\xi|\,d\xi=\vartheta(x,\mu),\quad |x|\leqslant 1 \qquad (1.1)$$

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*) Analogous problems on interaction of a stamp with an elastic half-plane were examined in a number of papers by other authors (see, for example, appropriate problems and their reviews in [2]).